

Boundary fermion currents and subleading order chiral anomaly in the  
AdS/CFT correspondence.

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**Abstract**

We construct a wave-functional whose argument couples to boundary fermion currents in the AdS/CFT correspondence. Using this we calculate the contributions from bulk fermions to the chiral anomaly that give the subleading order term in the exact  $N$ -dependence of the chiral anomaly of  $\mathcal{N} = 4$  SYM. The result agrees with the calculation of Bilal & Chu.

The AdS/CFT correspondence is typically studied in the large- $N$  limit, corresponding to the planar limit of the boundary gauge theory and the classical limit of the bulk supergravity/string theory. This limit allows us to calculate many interesting quantities on both sides of the correspondence to leading order in the large- $N$  expansion. Matching the results then gives us a dictionary that allows us to translate gauge theory effects into the language of holographic supergravity.

Although the leading order in  $N$  is extremely illuminating by itself, when we go beyond this approximation to consider quantum loops in AdS and subleading order effects in the gauge theory, the dictionary becomes much more subtle. This becomes clear when we consider the holographic origin of effects such as the Weyl and R-symmetry anomalies [4, 5]. At leading order these receive contributions only from the classical graviton and Chern-Simons actions, respectively, but at order  $1/N^2$  there are one-loop contributions from all the fields of supergravity.

One of the first calculations in the AdS/CFT correspondence of such an order  $1/N^2$  effect was the calculation of Bilal & Chu of order  $1/N^2$  corrections to the R-symmetry anomaly [8]. The purpose of this note is to show how the result of [8] arises in a wave-functional formalism. Our method involves the construction of a generating function for fermion current correlators that may be useful for more general calculations. In the usual analysis the subleading order corrections are induced by a Pauli-Villars regularisation that is introduced to regulate spinor loops in the bulk [10, 9]. Instead, we use heat-kernel methods to regulate the generating functional for boundary fermion currents.

In the Type IIB supergravity action the fermions couple to bulk vector fields that are dual to the boundary R-current. Treating these gauge vectors as background fields, a one-loop integration over the fermion fields generates an effective action depending on the gauge fields. This effective action includes Chern-Simons terms that contribute to the anomaly. If we interpret the radial coordinate of AdS as Euclidean "time", the cutoff near the boundary is a surface on which we can perform canonical quantisation. According to the standard prescription of AdS/CFT the Schrödinger wave-functional of supergravity fields defined on this surface is identified with the partition function of the boundary theory. This wave-functional includes radiatively induced Chern-Simons terms.

The full action for Type IIB Supergravity compactified on  $AdS_5 \times S^5$  has not been determined. However, the three-point couplings of fermion fields to gauge vectors that we need are obtained by covariantising the derivatives in the quadratic action. First, though, let us consider the quadratic action without couplings to gauge fields. The fermion action is

$$\int d^{d+1}x \sqrt{G} \bar{\psi} (\gamma^A D_A - m) \psi, \quad (1)$$

(capital Roman indices run from 0 to 4) and the bulk metric (incorporating a curved boundary) is

$$ds^2 = G_{AB} dX^A dX^B = dr^2 + z^{-2} e^\rho \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu, \quad e^{\rho/2} = 1 - C z^2, \quad C = \frac{l^2 \hat{R}}{4d(d-1)}, \quad (2)$$

where  $z = \exp(r/l)$  with  $l$  a length scale for AdS. We will interpret  $r$  as Euclidean time. The spin-covariant derivative is defined via the funfbein

$$V_0^\alpha = \frac{1}{z}\delta_0^\alpha, \quad V_\mu^\alpha = \frac{1}{z}e^{\rho/2}\tilde{V}_\mu^\alpha, \quad (3)$$

where  $\tilde{V}_\mu^\alpha$  is the vierbein for the boundary metric (Greek indices run from 1 to 4). Making the change of variables  $\psi = z^2 e^{-\rho} \tilde{\psi}$  causes the volume element in the path-integral to become the usual flat-space one, and the kinetic term in the action acquires the usual form. The action can be written

$$\int d^{d+1}x \tilde{\psi} \left( \gamma^0 \partial_0 + z e^{-\rho/2} \gamma^\mu \tilde{D}_\mu - m \right) \tilde{\psi}. \quad (4)$$

The  $D_\mu$  derivative is spin-covariant with respect to the boundary metric.

We impose the following boundary conditions on  $\tilde{\psi}$ :

$$Q_+ \tilde{\psi}(\tau, x) = u(x) = Q_+ u(x), \quad \tilde{\psi}^\dagger(\tau, x) Q_- = u^\dagger(x) = u^\dagger(x) Q_-, \quad (5)$$

for  $\tau = \exp(r_0/l)$  a small time cutoff on  $z$  near the boundary at  $z = 0$  and some local projection operators  $Q_\pm$ . The remaining projections are represented by functional differentiation. The partition function takes the form

$$\Psi[u, u^\dagger] = \exp[f + u^\dagger \Gamma u], \quad (6)$$

and the Schrödinger equation that it satisfies can be written

$$\frac{\partial}{\partial r_0} \Psi = - \int d^d x \left( u^\dagger Q_- + \frac{\delta}{\delta u} Q_+ \right) h \left( Q_+ u + Q_- \frac{\delta}{\delta u^\dagger} \right) \Psi, \quad (7)$$

where  $h = \tau e^{-\rho/2} \gamma^0 \gamma^\mu \tilde{D}_\mu - \gamma^0 m$ . Note that from the four-dimensional point of view  $\gamma^0$  is what we usually call  $\gamma^5$ . For the moment we choose  $Q_\pm$  to be  $Q_\pm = \frac{1}{2}(1 \pm \gamma^4)$ . Inserting this into (7) and (6) gives

$$\dot{\Gamma} = \tau e^{-\rho/2} D_4 - m - 2\tau e^{-\rho/2} \gamma^0 \gamma^i D_i \Gamma + \Gamma^2 (\tau e^{-\rho/2} D_4 + m) \quad (8)$$

while the free energy  $f$  satisfies

$$\dot{f} = \text{Tr} \left( Q_+ e^{-\rho/2} \gamma^0 \gamma^i D_i + Q_- \Gamma (\tau e^{-\rho/2} D_4 + m) \right). \quad (9)$$

Here the index  $i$  runs from 1 to 3. So that we can regulate the free energy with a heat-kernel, we expand  $\Gamma$  in terms of the positive-definite operator  $(\gamma^\mu \tilde{D}_\mu)^2$  (this is possible because of the underlying euclidean invariance):

$$\Gamma = \sum_{n=0}^{\infty} \gamma^0 (b_n(r_0) + c_n(r_0) \gamma^\mu \tilde{D}_\mu) (\gamma^\mu \tilde{D}_\mu)^{2n}. \quad (10)$$

Inserting into (8) gives a difference equation for the coefficients  $b_n$  and  $c_n$  with a unique solution. So, for example,  $b_0 = -1$  and all other coefficients vanish as  $r_0 \rightarrow -\infty$ . To regulate (9) we use a Seeley-de Witt expansion of the heat-kernel:

$$(\gamma \cdot \tilde{D})^{2n} \sim \left( -\frac{\partial}{\partial s} \right)^n \exp \left( -s(\gamma \cdot \tilde{D})^2 \right), \quad (11)$$

$$\exp(-s(\gamma \cdot \tilde{D})^2) = \int d^4x \sqrt{\hat{g}} \frac{1}{16\pi^2 s^2} (1 + s a_1(x) + s^2 a_2(x) + s^3 a_3(x) + \dots) \quad (12)$$

The contribution to (9) proportional to the  $a_2$  coefficient of  $(\gamma \cdot \tilde{D})^2$  is finite as  $s \rightarrow 0$  and  $r_0 \rightarrow -\infty$  and as discussed in [1, 2] it determines the Weyl anomaly. This gives the same result as the calculation in [1, 2], as it should for our choice of boundary conditions, according to the analysis of [3]. The above discussion extends trivially to boundary conditions given by

$$Q_+ = \frac{1}{2}(1 \pm \gamma^\mu). \quad (13)$$

We wish to study the effect of a chiral (R-symmetry) transformation of the fermion fields. To make the relation to fermion currents more explicit, we construct a wave-functional that corresponds to a generating functional of R-current correlators. The partition function (6) can be expressed as a path-integral over the upper half-plane cut off at  $z = \tau$  so that

$$\Psi[u^\dagger, u] = \int D\psi^\dagger D\psi e^{-S + \int d^4x (u^\dagger \psi - \psi^\dagger u)}, \quad (14)$$

where  $S$  is the action (4), and the boundary conditions are given by (5). Now consider

$$\Phi[\zeta] = \int Du^\dagger Du \Psi[\zeta u^\dagger, u] e^{\int d^4x u^\dagger \gamma^5 u} = \int D\psi^\dagger D\psi e^{-S + \int d^4x \zeta \psi^\dagger Q_+ \gamma^5 \psi}. \quad (15)$$

According to the standard AdS/CFT prescription this is the generating functional for correlators of the fermion current  $\psi^\dagger Q_+ \gamma^5 \psi$ . So we find

$$\begin{aligned} \langle \psi^\dagger Q_+ \gamma^5 \psi \rangle &= \left. \frac{\delta}{\delta \zeta} \det^{1/2}(\zeta \Gamma + \gamma^5) \right|_{\zeta=0} \\ &= \left. \frac{1}{2} \text{tr} \left( \frac{\Gamma}{\gamma^5 + \zeta \Gamma} \right) \right|_{\zeta=0} = \frac{1}{2} \text{tr}(Q_+ \gamma^5 \Gamma) \end{aligned} \quad (16)$$

Since this holds for all boundary conditions of the form (13) we conclude that

$$\langle \bar{\psi} \gamma^\mu \psi \rangle = \frac{1}{2} \text{tr}(\gamma^\mu \gamma^5 \Gamma) \quad (17)$$

This vanishes (as we expect) even when couplings to gauge fields are included in the heat-kernel regularisation of the trace.

The chiral anomaly can be obtained by considering the current  $\langle \psi^\dagger \psi \rangle$  that can be obtained in our formalism as follows.

At the cost of a spurious enlargement of the Hilbert space, we can represent fermion operators on the boundary by [11]

$$\psi^\dagger \sim \frac{1}{\sqrt{2}} \left( v^\dagger + \frac{\delta}{\delta v} \right), \quad \psi \sim \frac{1}{\sqrt{2}} \left( v + \frac{\delta}{\delta v^\dagger} \right). \quad (18)$$

When we construct solutions of the Schrödinger equation using the representation (18) they take the form

$$\Psi[v^\dagger, v] = \exp[f + 2v^\dagger(Q_+ - Q_- + Q_- \Gamma Q_+)v], \quad (19)$$

with  $\Gamma$  the same kernel as before, so it may seem as though not much has changed, but the difference is that  $v^\dagger$  and  $v$  are now unconstrained, and we can express

$$\langle \psi^\dagger \psi \rangle = \Phi_0[\zeta]|_{\zeta=0} \quad (20)$$

$$\Phi_0[\zeta] = \int Dv^\dagger Dv \Psi[\zeta v^\dagger, v] e^{\int d^4x 2v^\dagger v}. \quad (21)$$

Using the same boundary conditions as before, this gives

$$\langle \psi^\dagger \psi \rangle = \frac{1}{2} \text{tr}(Q_+ - Q_- + Q_- \Gamma Q_+) \sim \frac{1}{2} \text{tr}(Q_+ - Q_- - 2Q_- \gamma^5) = \frac{1}{2} \text{tr} \gamma^5. \quad (22)$$

It is useful to note that we get the same result if we use the boundary conditions  $Q_\pm = \frac{1}{2}(1 \pm \gamma^5)$  and the corresponding solution found in [2], since then  $\Gamma$  vanishes as  $\tau \rightarrow 0$ . The trace should be regulated with the heat-kernel expansion (12) of the operator  $(\gamma \cdot \tilde{D})^2$  (covariantised with respect to the gauge fields as well as the boundary metric). Turning on the background gauge fields is necessary to give a non-zero result, since there is no gravitational anomaly.

It is easy to check that the contribution of fermion fields to the anomaly given by (22) agrees with the calculation of [8], and therefore summing over the full Kaluza-Klein spectrum gives the exact N-dependence of the boundary chiral anomaly.

In conclusion, the subleading order part of the exact N-dependence of the boundary chiral anomaly coming from spinor loops in the bulk can be read off from the heat-kernel regularisation of the wave-functional (21). This is different from the usual method for obtaining the anomaly. The formalism may give a useful way of calculating more general correlators of fermion currents in the AdS/CFT correspondence.

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